数理解析研究所講究録 1287 巻 2002 年 127-142

Ergodic control problems arising from portfolio optimization for factor models

長井　英生

大阪大学大学院基礎工学研究科数理科学分野

HIDEO NAGAI

Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, 560-8531, Japan,
E-mail: nagai@sigmath.es.osaka-u.ac.jp

1 Introduction

Let us consider the following ergodic type Bellman equation of risk-sensitive control:

$$\chi=\frac{1}{2}\Delta v+\frac{\theta}{2}|\nabla v|^{2}+\inf_{z\in R^{N}}\{z^{*}\nabla v+z^{*}Ax+\frac{1}{2}|z|^{2}\}+V(x), \tag{1.1}$$

where χ is a constant and it is considered to characterize the minimum of the risk-sensitive long-run criterion:

$$\liminf_{T\to\infty}\frac{1}{\theta T}\log E_{x}[e^{\theta\int_{0}^{T}\{V(X_{s})+\frac{1}{2}|z_{s}|^{2}+z_{s}^{*}AX_{s}\}ds}]$$

subject to

$$dX_{t}=z_{t}dt+dW_{t}.$$

where W_t is an $N-$ dimensional standard Brownian motion process on a filtered probability space and z_t is a control process taking its value on R^N. Note that (1.1) is rewritten as

$$\chi=\frac{1}{2}\Delta v-Ax\cdot\nabla v-\frac{1-\theta}{2}|\nabla v|^{2}+V(x)-\frac{1}{2}x^{*}A^{*}Ax \tag{1.2}$$

and in a similar way to [15] we can see that there exists a solution of (1.1) such that $(1-\theta)v+\frac{1}{2}x^{*}Ax\to\infty$, $|x|\to\infty$, if A is symmetric and

$$\tilde{V}:=\frac{1}{2}|Ax|^{2}-(\theta-1)(V-\frac{1}{2}x^{*}A^{2}x)-\frac{1}{2}\mathrm{tr}A\to\infty,\quad |x|\to\infty. \tag{1.3}$$

Furthermore, the solution is represented as

$$v=-\frac{1}{1-\theta}\log\psi+\frac{1}{2(\theta-1)}x^{*}Ax$$

and

$$\chi=\frac{1}{1-\theta}\lambda_{1}+\frac{\min\tilde{V}}{1-\theta},$$

where ψ is the principal eigenfunction of the Schrödinger operator $-\frac{1}{2}\Delta + (\tilde{V} - \min \tilde{V})$ and λ_1 the corresponding eigenvalue :

$$-\frac{1}{2}\Delta\psi + (\tilde{V} - \min \tilde{V})\psi = \lambda_1 \psi$$

The infimum in (1.1) is attained by

$$z = -Ax - \nabla v(x)$$

so, taking a solution of

$$(1.4) \qquad d\hat{X}_t = (-A\hat{X}_t - \nabla v(\hat{X}_t))dt + dW_t, \quad X_0 = x$$

and defining a control $\hat{z}_t$ by

$$\hat{z}_t = -A\hat{X}_t - \nabla v(\hat{X}_t),$$

then we have

$$v(\hat{X}_t) - v(x) = \int_0^t \{(-A\hat{X}_t - \nabla v(\hat{X}_t)) \cdot \nabla v(\hat{X}_t) + \frac{1}{2}\Delta v(\hat{X}_t)\}dt + \int_0^t \nabla v(\hat{X}_t)dW_t.$$

Therefore, we see that

$$e^{\theta \int_0^T \{V(\hat{X}_s) + \frac{1}{2}|\hat{z}_s|^2 + \hat{z}_s^* A\hat{X}_s\}ds} = e^{\theta\chi T + \theta v(x) - \theta v(\hat{X}_T) + \theta \int_0^T \nabla v(\hat{X}_s)dW_s - \frac{\theta^2}{2}\int_0^T |\nabla v|^2(\hat{X}_s)ds}.$$

When introducing a probability measure $\hat{P}$ by

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_T} = e^{\theta \int_0^T \nabla v(\hat{X}_s)dW_s - \frac{\theta^2}{2}\int_0^T |\nabla v|^2(\hat{X}_s)ds}$$

we see that

$$E_x[e^{\theta \int_0^T \{V(\hat{X}) + \frac{1}{2}|\hat{z}_s|^2 + \hat{z}_s^* AX_s\}ds}] = e^{\chi\theta T + \theta v(x)} \hat{E}_x[e^{-\theta v(\hat{X}_T)}].$$

By using a new Brownian motion $\hat{W}_t$ under $\hat{P}$, we can rewrite (1.4) as

$$\begin{aligned} d\hat{X}_t &= (-A\hat{X}_t - (1-\theta)\nabla v(\hat{X}_t))dt + d\hat{W}_t \\ &= \nabla \log \psi(\hat{X}_t)dt + d\hat{W}_t. \end{aligned}$$

Note that $(\hat{X}_t, \hat{P}_x)$ ia an ergodic diffusion process with an invariant measure $\psi(x)^2 dx$. If

$$(1.5) \qquad \hat{E}_x[e^{-\theta v(\hat{X}_T)}] \to \int e^{-\theta v(x)}\psi(x)^2 dx < \infty$$

as $T \to \infty$, then

$$\frac{1}{\theta T} \log E_x[e^{\theta \int_0^T \{V(\hat{X}) + \frac{1}{2}|\hat{z}_s|^2 + \hat{z}_s^* AX_s\}ds}] \to \chi,$$

which indicates that $\hat{z}_t$ is an optimal strategy. However it is not always the case. Indeed we can give an example where (1.5) could be violated in what follows.

We consider the case where $N = 1$, $\theta > 0$ and $V(x) = \frac{c}{2}x^2$, $c > 0$. Then, $\tilde{V} = \frac{1}{2}A^2x^2 - (\theta-1)(V - \frac{1}{2}A^2x^2) - \frac{1}{2}A = \frac{1}{2}\theta A^2x^2 - \frac{\theta-1}{2}cx^2 - \frac{1}{2}A$. Therefore, if

$$\theta A^2 - (\theta - 1)c > 0, \tag{1.6}$$

then (1.3) holds. Under this assumption we see that $\psi(x) = e^{-\frac{1}{2}gx^2}$, where $g = \sqrt{\theta A^2 - (\theta-1)c}$. Since

$$v = -\frac{1}{1-\theta}\log\psi + \frac{1}{2\theta-1}Ax^2 = \frac{1}{2(1-\theta)}(g-A)x^2$$

condition (1.5) reads

$$\int e^{-\frac{\theta}{2(1-\theta)}(g-A)x^2}e^{-gx^2}dx < \infty. \tag{1.5'}$$

Thus, we need check whether $\frac{-\theta}{2(1-\theta)}(g-A) - g < 0$ holds or not. As a result we see that, if i) $1 < \theta \leq 4$, $A < 0$ or ii) $4 < \theta$, $A < 0$, $\frac{\theta(\theta-4)}{(\theta-2)^2}A^2 \leq c < \frac{\theta}{\theta-1}A^2$, then (1.5)' is violated. Otherwise it holds under (1.6).

Such a situation occurs in discussing ergodic control problems with criteria of exponential type and it is Fleming and Sheu that has noticed first by taking up one dimensional problems concerning risk-sensitive portofolio optimizaion (cf. Fleming and Sheu [10]). Related problems have been discussed extensively in [4],[5],[11],[12],[18],[21].

In the present paper, by taking up risk-sensitive portfolio optimization problems for general factor models, we shall consider constructing optimal strategies for the problems on infinite time horizon by using the solutions of corresponding ergodic type Bellman equations. We shall show that the solutions define optimal strategies under some condition which suggest an integrability condition such as (1.5) by the invariant measures of underlying ergodic diffusion processes. The ergodic diffusion processes are the optimal ones of some other classical ergodic control problems with the same Bellman equations of ergodic type, which correspond to the diffusion process $(\hat{X}_t, \hat{P}_x)$ in the case of the above example.

2 Finite time horizon case

We consider a market with $m+1 \geq 2$ securities and $n \geq 1$ factors. We assume that the set of securities includes one bond, whose price is defined by ordinary differential equation:

$$dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0, \tag{2.1}$$

where $r(x)$ is a nonnegative bounded function. The other secutity prices S^i_t, $i = 1, 2, ..., m$ and factors X_t are assumed to satsfy the following stochastic differential equations:

$$\begin{aligned} &dS^i(t) = S^i(t)\{g^i(X_t)dt + \textstyle\sum_{k=1}^{n+m}\sigma^i_k(X_t)dW^k_t\}, \\ &S^i(0) = s^i, \quad i = 1, ..., m \end{aligned} \tag{2.2}$$

$$\begin{aligned}
&dX_t = b(X_t)dt + \lambda(X_t)dW_t, \\
&X(0) = x \in R^n,
\end{aligned} \tag{2.3}$$

where $W_t = (W_t^k)_{k=1,..,(n+m)}$ is a $m+n$ dimensional standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. Here σ and λ are respectively $m \times (m+n)$,$n \times (m+n)$ matrix valued functions. We assume that

$$\begin{aligned}
&g,\ \sigma,\ b,\ \lambda \ \text{ are locally Lipshitz} \\
&c_1|\xi|^2 \le \xi^* \sigma\sigma^*(x)\xi \le c_2|\xi|^2, \quad c_1, c_2 > 0 \\
&|g(x)| \le K(1+|x|) \\
&x^* b(x) + \tfrac{1}{2} \parallel \lambda\lambda^*(x) \parallel \le K(1+|x|^2)
\end{aligned} \tag{2.4}$$

where σ^* stands for the transposed matrix of σ.

Let us denote investment strategy to i-th security $S^i(t)$ by $h^i(t)$, $i = 0, 1, ..., m$ and set

$$S(t) = (S^1(t), S^2(t), ..., S^m(t))^*,$$

$$h(t) = (h^1(t), h^2(t), ...,, h^m(t))^*$$

and

$$\mathcal{G}_t = \sigma(S(u), X(u); u \le t).$$

Here S^* stands for transposed matrix of S.

Definition 2.1 $(h^0(t), h(t)^*)_{0\le t\le T}$ *is said an invetment strategy if the following conditions are satisfied*

i) $h(t)$ *is a* R^m *valued* $\mathcal{G}_t$ *progressively measurable stochastic process such that*

$$\sum_{i=1}^{m} h^i(t) + h^0(t) = 1 \tag{2.5}$$

ii)

$$P(\int_0^T |h(s)|^2 ds < \infty) = 1.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. When $(h^0(t), h(t)^*)_{0\le t\le T} \in \mathcal{H}(T)$ we will often write $h \in \mathcal{H}(T)$ for simplicity since h^0 is determined by (2.5).

For given $h \in \mathcal{H}(T)$ the process $V_t = V_t(h)$ representing the investor's capital at time t is determined by the stochastic differential equation:

$$\begin{aligned}
\frac{dV_t}{V_t} &= \textstyle\sum_{i=0}^m h^i(t)\frac{dS^i(t)}{S^i(t)} \\
&= h^0(t)r(X_t)dt + \textstyle\sum_{i=1}^m h^i(t)\{g^i(X_t)dt + \sum_{k=1}^{m+n} \sigma_k^i(X_t)dW_t^k\} \\
V_0 &= v.
\end{aligned}$$

Then, taking (2.5) into account it turns out to be a solution of

$$\begin{aligned} \frac{dV_t}{V_t} &= r(X_t)dt + h(t)^*(g(X_t) - r(X_t)\mathbf{1})dt + h(t)^*\sigma(X_t)dW_t, \\ V_0 &= v, \end{aligned} \tag{2.6}$$

where $\mathbf{1} = (1, 1, ..., 1)^*$.

We first consider the following problem. For a given constant $\theta > -2$, $\theta \neq 0$ maximize the following risk-sensitized expected growth rate up to time horizon T:

$$J(v, x; h; T) = -\frac{2}{\theta} \log E[e^{-\frac{\theta}{2}\log V_T(h)}], \tag{2.7}$$

where h ranges over the set $\mathcal{A}(T)$ of all admissible strategies defined later. Then we consider the problem of maximizing the risk-sensitized expected growth rate per unit time

$$J(v, x; h) = \limsup_{T\to\infty} (\frac{-2}{\theta T}) \log E[e^{-\frac{\theta}{2}\log V_T(h)}], \tag{2.8}$$

where h ranges over the set of all investment straregies such that $h \in \mathcal{A}(T)$ for each T.

Since V_t satisfies (2.6) we have

$$\begin{aligned} V_t^{-\theta/2} &= v^{-\theta/2}\exp\{\tfrac{\theta}{2}\textstyle\int_0^t \eta(X_s, h_s))ds \\ &- \tfrac{\theta}{2}\textstyle\int_0^t h_s^*\sigma(X_s)dW_s - \tfrac{\theta^2}{8}\int_0^t h_s^*\sigma\sigma^*(X_s)h_s ds\}, \end{aligned}$$

where

$$\eta(x, h) = (\frac{\theta+2}{4})h^*\sigma\sigma^*(x)h - r(x) - h^*(g(x) - r(x)\mathbf{1}).$$

If a given investment strategy h satisfies

$$E[e^{-\frac{\theta}{2}\int_0^T h(s)^*\sigma^*(X_s)dW_s - \frac{\theta^2}{8}\int_0^T h(s)^*\sigma\sigma^*(X_s)h(s)ds}] = 1, \tag{2.9}$$

then we can introduce a probability measure P^h given by

$$P^h(A) = E[e^{-\frac{\theta}{2}\int_0^T h^*(s)\sigma(X_s)dW_s - \frac{\theta^2}{8}\int_0^T h^*(s)\sigma\sigma^*(X_s)h(s)ds}; A]$$

for $A \in \mathcal{F}_T$, $T > 0$. By the probability measure P^h our criterion $J(v, x; h; T)$ and $J(v, x; h)$ can be written as follows:

$$J(v, x; h, T) = \log v - \frac{2}{\theta} \log E^h[e^{\frac{\theta}{2}\int_0^T \eta(X_s, h(s))ds}] \tag{2.7'}$$

$$(2.8)' \qquad J(v,x;h) = \liminf_{T\to\infty} -\frac{2}{\theta T}\log E^h[e^{\frac{\theta}{2}\int_0^T \eta(X_s,h(s))ds}].$$

On the other hand, under the probability measure,

$$\begin{aligned} W_t^h &= W_t - \langle W_\cdot, -\tfrac{\theta}{2}\int_0^\cdot h^*(s)\sigma(X_s)dW_s\rangle_t \\ &= W_t + \tfrac{\theta}{2}\int_0^t \sigma^*(X_s)h(s)ds \end{aligned}$$

is a standard Brownian motion process, and therefore, the factor process X_t satisfies the following stochastic differential equation

$$(2.10) \qquad dX_s = (b(X_s) - \frac{\theta}{2}\lambda\sigma^*(X_s)h(s))ds + \lambda(X_s)dW_s^h.$$

We regard (2.10) as a stochastic differential equation controlled by h and the criterion function is written by P^h as follows:

$$(2.11) \qquad J(v,x;h;T-t) = \log v - \frac{2}{\theta}\log E^h[e^{\frac{\theta}{2}\int_0^{T-t}\eta(X_s,h(s))ds}]$$

and the value function

$$(2.12) \qquad u(t,x) = \sup_{h\in\mathcal{H}(T-t)} J(v,x;h;T-t), \ 0\le t\le T.$$

Then, according to Bellman's dynamic programming principle, it should satisfy the following Bellman equation

$$(2.13) \qquad \begin{aligned} &\frac{\partial u}{\partial t} + \sup_{h\in R^m} L^h u = 0, \\ &u(T,x) = \log v, \end{aligned}$$

where L^h is defined by

$$L^h u(t,x) = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2u) + (b(x) - \frac{\theta}{2}\lambda\sigma^*(x)h)^*Du - \frac{\theta}{4}(Du)^*\lambda\lambda^*(x)Du - \eta(x,h).$$

Note that $\sup_{h\in R^m} L^h u$ can be written as

$$\begin{aligned} \sup_{h\in R^m} L^h u(t,x) &= \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2u) + (b - \frac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(g-r\mathbf{1}))^*Du \\ &-\tfrac{\theta}{4}(Du)^*\lambda(I - \tfrac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma)\lambda^*Du + \tfrac{1}{\theta+2}(g-r\mathbf{1})^*(\sigma\sigma^*)^{-1}(g-r\mathbf{1}) \end{aligned}$$

Therefore our Bellman equation (2.13) is written as follows:

$$\text{(2.14)}\qquad \begin{aligned}&\tfrac{\partial u}{\partial t} + \tfrac{1}{2}\mathrm{tr}(\lambda\lambda^* D^2 u) + B(x)^* Du - (Du)^*\lambda N^{-1}\lambda^* Du + U(x) = 0,\\ &u(T,x) = \log v,\end{aligned}$$

where

$$\text{(2.15)}\qquad \begin{aligned}B(x) &= b(x) - \tfrac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(g(x) - r(x)\mathbf{1})\\ N^{-1}(x) &= \tfrac{\theta}{4}(I - \tfrac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma(x))\\ U(x) &= \tfrac{1}{\theta+2}(g - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(g - r\mathbf{1}).\end{aligned}$$

As for (2.14) we note that if $\theta > 0$, then

$$\frac{\theta}{2(\theta+2)} I \le N^{-1} \le \frac{\theta}{4} I$$

and therefore we have

$$-\frac{\theta}{4}\lambda\lambda^* \le -\lambda N^{-1}\lambda^* \le -\frac{\theta}{2(\theta+2)}\lambda\lambda^*.$$

Such kinds of equations have been studied in Nagai [20], or Bensoussan, Frehse and Nagai [3]. Here we can obtain the following result along the line of [3], Theorem 5.1 with refinement on estimate (2.17).

Theorem 2.1 *i) If, in addition to (2.4), $\theta > 0$ and*

$$\text{(2.16)}\qquad \nu_r|\xi|^2 \le \xi^*\lambda\lambda^*(x)\xi \le \mu_r|\xi|^2,\ \ r = |x|,\ \ \nu_r,\ \mu_r > 0,$$

then we have a solution of (2.14) such that

$$\begin{aligned}&u,\ \tfrac{\partial u}{\partial t},\ D_k u,\ D_{kj}u \in L^p(0,T; L^p_{loc}(R^n)), \quad 1 < \forall p < \infty\\ &\tfrac{\partial^2 u}{\partial t^2},\ \tfrac{\partial D_k u}{\partial t},\ \tfrac{\partial D_{kj}u}{\partial t}, D_{kjl}u \in L^p(0,T; L^p_{loc}(R^n)), \quad 1 < \forall p < \infty\\ &u \ge \log v,\ \ \tfrac{\partial u}{\partial t} \le 0.\end{aligned}$$

Furthermore we have the estimate

$$\text{(2.17)}\qquad \begin{aligned}|\nabla u|^2(t,x) - \frac{c_0}{\nu_r}\frac{\partial u}{\partial t}(t,x) &\le c_r(|\nabla Q|^2_{2r} + |Q|^2_{2r} + |\nabla(\lambda\lambda^*)|^2_{2r}\\ &+ |\nabla B|_{2r} + |B|^2_{2r} + |U|_{2r} + |\nabla U|^2_{2r} + 1), \qquad x \in B_r,\ t \in [0,T)\end{aligned}$$

where

$$\begin{aligned}&Q = \lambda N^{-1}\lambda^*,\ \ c_0 = \tfrac{4(1+c)(\theta+2)}{\theta},\ \ c > 0\\ &|\cdot|_{2r} = \|\cdot\|_{L^\infty(B_{2r})}\end{aligned}$$

and c_r is a positive constant depending on n, r, ν_r, μ_r and c.

ii) If, in addition to the above conditions,

$$\inf_{|x|\geq r} U(x) \to \infty, \quad as \ r \to \infty,$$

then the above solution u satisfies

$$\inf_{|x|\geq r, t\in(0,T)} u(x,t) \to \infty, \quad as \ r \to \infty.$$

Moreover, there exists at most one such solution in $L^\infty(0,T;W^{1,\infty}_{loc}(R^n))$

Remark. If

$$\frac{1}{\nu_r},\ \mu_r \leq M(1+r^m), \quad \exists m > 0, \tag{2.21}$$

then we have

$$c_r \leq M'(1+r^{m'}), \ \exists m'$$

in estimete (2.17). In particular, if $m = 0$, then c_r can be taken independent of r.

Let us define a class of admissible investment strategy $\mathcal{A}_T$ as the set of investment strategies satisfying (2.9). Then, thanks to the above theorem and remark we have the following proposition.

Proposition 2.1 *i) We assume the assumptions in the above theorem and let u be a solution of (2.14). Define*

$$\hat{h}_t = \hat{h}(t, X_t)$$
$$\hat{h}(t,x) = \tfrac{2}{\theta+2}(\sigma\sigma^*)^{-1}(g - r\mathbf{1} - \tfrac{\theta}{2}\sigma\lambda^* Du)(t,x),$$

where X_t is the solution of (2.3), then, under the assumption that

$$E[e^{-\int_0^T(2N^{-1}\lambda^* Du+\theta K)^*(x_s)dW_s - \frac{1}{2}\int_0^T(2N^{-1}\lambda^* Du+\theta K)^*(2N^{-1}\lambda^* Du+\theta K)(x_s)ds}] = 1, \tag{2.22}$$

where

$$K = \frac{1}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}(g - r\mathbf{1}),$$

$\hat{h}_t \in \mathcal{A}_T$ is an optimal strategy for the portfolio optimization problem of maximizing the criterion (2.7).

ii) if

$$\begin{gathered} c_1|\xi|^2 \leq \xi^*\lambda\lambda^*(x)\xi \leq c_2|\xi|^2, \ c_1, c_2 > 0 \\ g,\ b,\ \lambda,\ \sigma \ \textit{are globally Lipshitz,} \end{gathered} \tag{2.23}$$

then (2.22) is valid.

To disccuss the problem on infinite time horizon we introduce another stochastic control problem on a finite time horizon with the same Bellman equation as (2.14) and then consider its ergodic counter part. For that let us set

$$G = b - \lambda\sigma^*(\sigma\sigma^*)^{-1}(g - r\mathbf{1})$$

and rewrite equation (2.14) as

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2u) + G(x)^*Du \\
(2.24)\qquad &\qquad - (-\lambda^*Du + NK)^*N^{-1}(-\lambda^*Du + NK)(x) + \tfrac{\theta+2}{2}K^*NK(x) = 0, \\
&u(T,x) = \log v.
\end{aligned}$$

Since

$$-(-\lambda^*Du + NK)^*N^{-1}(-\lambda^*Du + NK) = \inf_{z\in R^{n+m}}\{z^*Nz + 2z^*NK - 2(\lambda z)^*Du\},$$

we can regard (2.21) as the Bellman equation of the following stochastic control peoblem. Set

$$(2.25)\quad u(t,x) = \inf_{Z.} E_x\Big[\int_0^{T-t}\{Z_s^*N(Y_s)Z_s + 2Z_s^*NK(Y_s) + \frac{\theta+2}{2}K^*NK(Y_s)\}ds + \log v\Big],$$

where Y_t is a controlled process governed by the stochastic differential equatinion

$$(2.26)\qquad dY_t = \lambda(Y_t)dW_t + (G(Y_t) - 2\lambda(Y_t)Z_t)dt,$$

and Z_t is a control taking its value on R^{n+m}. We define the set of admissible controls Z_t as all progressively measurable processes satisfying

$$E_x\Big[\int_0^T |Z_s|^{2q}ds\Big] < \infty, \quad \forall q \geq 1.$$

An ergodic counterpart of the above problem is formulated as follows. Consider the problem:

$$(2.27)\qquad \chi = \inf_{Z.}\liminf_{T\to\infty}\frac{1}{T}E_x\Big[\int_0^T\{Z_s^*N(Y_s)Z_s + 2Z_s^*NK(Y_s) + \frac{\theta}{2}K^*NK(Y_s)\}ds\Big]$$

with controlled process Y_t governed by (2.26). Then, corresponding Bellman equation is written as

$$\begin{aligned}
&\chi = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2w) + G(x)^*Dw \\
(2.28)\qquad &\qquad - (-\lambda^*Dw + NK)^*N^{-1}(-\lambda^*Dw + NK)(x) + \tfrac{\theta+2}{2}K^*NK(x),
\end{aligned}$$

whose original one is

$$\chi = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2w) + B(x)^*Dw - (Dw)^*\lambda N^{-1}\lambda^*(x)Dw + U(x) = 0, \tag{2.29}$$

namely,

$$\chi = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2w) + (b - \frac{\theta}{\theta+2}\lambda\sigma^*(\sigma\sigma^*)^{-1}(g - r\mathbf{1}))^*Dw$$
$$- \tfrac{\theta}{4}(Dw)^*\lambda(I - \tfrac{\theta}{\theta+2}\sigma^*(\sigma\sigma^*)^{-1}\sigma)\lambda^*Dw + \tfrac{1}{\theta+2}(g - r\mathbf{1})^*(\sigma\sigma^*)^{-1}(g - r\mathbf{1}).$$

In the following section we shall analyze the Bellman equation of ergodic type (2.28). Indeed we shall deduce equation (2.28), accordingly (2.29), as the limit of parabolic type equation (2.24) as $T \to \infty$ under suitable conditions.

Remark. To regard our Bellman equation as (2.24) has a meaning from financial view points. Indeed, under the minimal martingale measure $\tilde{P}$ (cf. [7] Proposition 1.8.2 as for minimal martingale measures), which is defined by

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_T} = e^{-\int_0^T \zeta(X_s)^* dW_s - \frac{1}{2}\int_0^T |\zeta(X_s)|^2 ds},$$

$\zeta(x) = \sigma^*(\sigma\sigma^*)^{-1}(x)(g(x) - r(x)\mathbf{1})$ factor process X_t is the diffusion process with the genarator

$$L = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2) + G(x)^*D,$$

namely, it is governed by the SDE

$$dX_t = \lambda(X_t)d\tilde{W}_t + G(X_t)dt$$

Here $\tilde{W}_t = W_t + \int_0^t \zeta(X_s)ds$ and it is a brownian motion under the probability measure $\tilde{P}$.

3 Ergodic type Bellman equation

In what follows we assume that

$$\frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)) + x^*G(x) + \frac{\kappa}{2}\frac{x^*\lambda\lambda^*(x)x}{\sqrt{1+|x|^2}} \le 0, \qquad |x| \ge \exists r > 0,\ \kappa > 0 \tag{3.1}$$

and set

$$L = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2) + G^*(x)D.$$

Proposition 3.1 *We assume (2.4), (3.1) and (2.16) with*

$$\nu_r \ge e^{-\frac{\kappa-c}{8}r}, \qquad r >> 1,\ c > 0, \tag{3.2}$$

then L diffusion process $(\tilde{P}_x, X_t)$ is ergodic and satisfies

$$\tilde{E}_x[e^{\kappa\sqrt{1+|X_t|^2}}] \le e^{\kappa\sqrt{1+|x|^2}} \tag{3.3}$$

Theorem 3.1 *Assume the assumptions of Theorem 2.1, (3.1) and that*

$$\frac{1}{\nu_r},\ \mu_r \leq K(1+r^m)$$
$$|Q|,\ |\nabla Q|,\ |B|,\ |\nabla B|,\ U,\ |\nabla U|,\ |\nabla(\lambda\lambda^*)| \leq K(1+|x|^m),$$

then, as $T \to \infty$,

$$u(0,x;T) - u(0,0;T) \to w(x),$$
$$\frac{1}{T}u(0,x;T) \to \chi,$$

uniformly on each compact set, where (w,χ) *is the solution of (2.28) such that* $w \in C^2(R^n)$.

Our Bellman equation of ergodic type (2.28) is rewritten as

$$\begin{aligned} \chi = &\frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2w) + G(x)^*Dw \\ &\inf_{z\in R^{n+m}}\{z^*Nz + 2z^*NK - 2(\lambda z)^*Dw\} + \tfrac{\theta+2}{2}K^*NK(x), \end{aligned} \tag{3.10}$$

and the infimum is attained by

$$\hat{z}(x) = N^{-1}\lambda^*(x)Dw(x) - K(x),$$

which define the following elliptic operator considered as the generator of the optimal diffusion for (2.27)

$$\hat{L} = \frac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2) + G^*(x)D - 2(\lambda N^{-1}\lambda^*(x)Dw(x) - \lambda K(x))^*D.$$

Then we have the following proposition.

Proposition 3.2 *Unde the assumption of Theorem 3.1* $\hat{L}$ *diffusion process is ergodic.*

4 Optimal strategy for portfolio optimization on infinite time horizon

Define the set of admissible strategies $\mathcal{A}$ by

$$\mathcal{A} = \{h : h \in \mathcal{A}(T), \forall T\}$$

and set

$$\hat{H}_t = \hat{H}(X_t)$$
$$\hat{H}(x) = \tfrac{2}{\theta+2}(\sigma\sigma^*)^{-1}(g - r\mathbf{1} - \tfrac{\theta}{2}\sigma\lambda^*Dw)(x),$$

where X_t is the solution of SDE (2.3), then we have the following theorem.

Theorem 4.1 *In addition to the assumptions of Theorem 3.1 we assume (2.23) and that*

$$\text{(4.1)}\qquad \frac{4}{\theta^2}(g-r\mathbf{1})^*(\sigma\sigma)^{-1}(g-r\mathbf{1})-(Dw)^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^*Dw\to\infty,\ \ |x|\to\infty,$$

then $\hat{H}_t$ is an optimal strategy for portfolio optimization maximizing long run criterion (2.8)

$$J(v,x;\hat{H})=\sup_{h\in\mathcal{A}}J(v,x;h).$$

Remark. Under the probability measure $\hat{P}_x$ the factor process is an ergodic diffusion process with the generator $\hat{L}$. In fact, by calculation, we can see that

$$\begin{aligned}&\tfrac{1}{2}\mathrm{tr}(\lambda\lambda^*D^2)+(b-\tfrac{\theta}{2}\lambda\sigma^*\hat{H}-\tfrac{\theta}{2}\lambda\lambda^*Dw)^*D\\&=\tfrac{1}{2}\mathrm{tr}(\lambda\lambda^*(x)D^2)+G^*(x)D-2(\lambda N^{-1}\lambda^*(x)Dw(x)-\lambda K(x))^*D.\end{aligned}$$

Then, under assumption (4.1), $\hat{L}$ diffusion process $(\hat{P}_x, X_t)$ satisfies

$$\hat{E}_x[e^{\frac{\theta}{2}w(X_T)}]\to\int e^{\frac{\theta}{2}w(x)}\mu(dx)<\infty,\quad \text{as } T\to\infty$$

where μ is the invariant measure of (P_x, X_t).

5 Example

Example (Linear Gaussian case)
Let us consider the case where

$$g(x)=a+Ax,\ \sigma(x)=\Sigma,$$
$$b(x)=b+Bx,\ \lambda(x)=\Lambda,$$
$$r(x)=r,$$

where $A,\ B,\ \Sigma,\ \Lambda$ are all constant matrices and a and b are constant vectors. Such a case has been considered by Bielecki and Pliska [4], [5],Fleming and Sheu [11],[12] and Kuroda and Nagai [18].

In this case the solution $u(t,x)$ of (2.14) has the following explicit form

$$u(t,x)=\frac{1}{2}x^*P(t)x+q(t)^*x+k(t)$$

where $P(t)$ is a solution of the Riccati differential equation:

$$\text{(5.1)}\qquad \begin{aligned}&\dot{P}(t)-P(t)K_0P(t)+K_1^*P(t)+P(t)K_1+\tfrac{2}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A=0,\\&P(T)=0,\end{aligned}$$

and

$$K_0 = \tfrac{\theta}{2}\Lambda(I - \tfrac{\theta}{\theta+2}\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^*$$
$$K_1 = B - \tfrac{\theta}{\theta+2}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A.$$

The term $q(t)$ is a solution of linear differential equation:

$$\dot{q}(t) + (K_1^* - P(t)K_0)q(t) + P(t)b + (\tfrac{2}{\theta+2}A^* - \tfrac{\theta}{\theta+2}P(t)\Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) = 0$$
$$q(T) = 0$$

and $k(t)$ a solution of

$$\dot{k}(t) + \frac{1}{2}\mathrm{tr}(\Lambda\Lambda^*P(t)) - \frac{\theta}{4}q(t)^*\Lambda\Lambda^*q(t) + b^*q(t) + r + \frac{1}{\theta+2}(a - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1})$$
$$+\tfrac{\theta^2}{4(\theta+2)}q(t)^*\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*q(t) - \tfrac{\theta}{\theta+2}(a - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*q(t) = 0$$
$$k(T) = \log v$$

If

$$G \equiv B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A \quad \text{is stable,}$$

then
i) $P(0) = P(0;T)$ converges, as $T \to \infty$, to a nonnegative definite matrix $\tilde{P}$, which is a solution of algebraic Riccati equation:

$$K_1^*\tilde{P} + \tilde{P}K_1 - \tilde{P}K_0\tilde{P} + \frac{2}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0.$$

Moreover $\tilde{P}$ satisfies the estimate

(5.2)
$$0 \le \tilde{P} \le \frac{2}{\theta}\int_0^\infty e^{sG^*}A^*(\Sigma\Sigma^*)^{-1}Ae^{sG}ds.$$

ii) $q(0) = q(0;T)$ converges, as $T \to \infty$, to a constant vector $\tilde{q}$, which satisfies

$$(K_1^* - \tilde{P}K_0)\tilde{q} + \tilde{P}b + (\frac{2}{\theta+2}A^* - \frac{\theta}{\theta+2}\tilde{P}\Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) = 0$$

iii) $\frac{k(0;T)}{T}$ converges to a constant $\rho(\theta)$ defined by

$$\rho(\theta) = \frac{1}{2}\mathrm{tr}(\tilde{P}\Lambda\Lambda^*) - \frac{\theta}{4}\tilde{q}^*\Lambda\Lambda^*\tilde{q} + b^*\tilde{q} + r + \frac{1}{\theta+2}(a - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1})$$
$$+\tfrac{\theta^2}{4\theta+8}\tilde{q}^*\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*\tilde{q} - \tfrac{\theta}{\theta+2}(a - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*\tilde{q}$$

if, moreover,

(5.3)
$$(B^*, A^*(\Sigma\Sigma^*)^{-1}\Sigma) \text{ is controllable,}$$

iv) the solution $\tilde{P}$ of the above algebraic Riccati equation is strictly positive definite.

Finally, if, in addition to the above conditions,

$$(5.4) \qquad (B, \Lambda) \text{ is controllable,}$$

then

v) the investment strategy $\tilde{h}_t$ defined by

$$\tilde{h}_t = \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}[a - r\mathbf{1} - \frac{\theta}{2}\Sigma\Lambda^*\tilde{q} + (A - \frac{\theta}{2}\Sigma\Lambda^*\tilde{P})X_t]$$

is optimal for the portofolio optimization on infinite time horizon maximizing the criterion (2.8):

$$\sup_{h\in\mathcal{A}} J(v, x; h) = J(v, x; \tilde{h}_.) = \rho(\theta)$$

if and only if

$$(5.5) \qquad \hat{P}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*\hat{P} < A^*(\Sigma\Sigma^*)^{-1}A,$$

where $\hat{P} = \frac{\theta}{2}\tilde{P}$ (cf. [18]).

Set

$$w(x) = \frac{1}{2}x^*\tilde{P}x + \tilde{q}^*x,$$

then $w(x)$ satisfies (2.28) and (5.5) is equivalent to

$$\int e^{\frac{\theta}{2}w(x)}\mu(dx) < \infty$$

under the assumptions (5.3) and (5.4), where $\mu(dx)$ is the invariant measure of $\hat{L}$ diffusion process. We consider the case where $n = m = 1$. Then $\Sigma\Sigma^*$, $\Lambda\Sigma^*$, A, B are all scalars and (5.5) is written as

$$(5.5') \qquad \frac{\theta^2}{4}\tilde{P}^2(\Lambda\Sigma^*)^2 < A^2$$

We can find sufficient condition for (5.5') by using estimate (5.2). Indeed, If

$$(5.6) \qquad A^2(\Lambda\Sigma^*)^2(\Sigma\Sigma^*)^{-2}(\int_0^\infty e^{2sG}ds)^2 < 1$$

then (5.5') holds. (5.6) is equivalent to

$$(2B(\Sigma\Sigma^*) - 3(\Lambda\Sigma^*)A)(2B(\Sigma\Sigma^*) - (\Lambda\Sigma^*)A) > 0,$$

from which we see that

$$(5.7) \qquad B < \frac{1}{2}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A \quad \text{if} \quad \Lambda\Sigma^*A > 0$$

$$B < \frac{3}{2}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A \quad \text{if} \quad \Lambda\Sigma^* A < 0 \tag{5.8}$$

since $G = B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A < 0$ by the stability assumption.

We illustrate an example where (5.5') is violated as follows. Set $\theta = 4$ and $B = \frac{2}{3}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A$, then we have

$$\tilde{P}^2(6\Lambda\Lambda^* - 4(\Lambda\Sigma^*)^2) = A^2$$

and therefore (5.5') is violated if and only if

$$6\Lambda\Lambda^* - 4(\Lambda\Sigma^*)^2 \le 4(\Lambda\Sigma^*)^2,$$

namely

$$4(\Lambda\Sigma^*)^2 \ge 3\Lambda\Lambda\Sigma\Sigma^*. \tag{5.9}$$

Set $\Lambda = (1, \lambda)$, $\Sigma = (1, \sigma)$, then (5.9) is equivalent to

$$\{\lambda\sigma + 1 + \sqrt{3}(\lambda - \sigma)\}\{\lambda\sigma + 1 - \sqrt{3}(\lambda - \sigma)\} \ge 0.$$

References

[1] R.N. Bhattacharya, "Criteria for recurrence and existence of invariant measures for multidimensional diffusions", The Annals of Probability, vol.6 (1978) 541-553

[2] A. Bensoussan, *Stochastic Control of Partially Observable Systems*, Cambridge (1992)

[3] A. Bensoussan, J. Frehse and H. Nagai, " Some Results on Risk-sensitive with full observation", Appl. Math. and its Optimization, vol.37 (1998)1-41

[4] T.R. Bielecki and S.R. Pliska, "Risk-Sensitive Dynamic Asset Management",*Appl. Math. Optim.* vol. 39 (1999) 337-360

[5] T.R. Bielecki and S.R. Pliska, "Risk-Sensitive Intertemporal CAPM, With Application to Fixed Income Management", preprint

[6] R.S. Bucy and P.D. Joseph, *Filtering for stochastic processes with applications to guidance*, Chelsea, New York (1987)

[7] El karoui, N. and Quenez, M-C. (1995) "Dynamic Programming pricing of contingent claims in an incomplete market", SIAM J. Cont. Optim. **33** 29-66

[8] W.H. Fleming and W.M. McEneaney, Risk-sensitive control on an infinite horizon, *SIAM J. Control and Optimization*, Vol. 33, No. 6 (1995) 1881-1915

[9] W.H. Fleming and R. Rishel, *Optimal Deterministic and Stochastic Control*, Springer-Verlag, Berlin (1975)

[10] W.H. Fleming and S.J. Sheu, "Optimal Long Term Growth Rate of Expected Utility of Wealth", Ann. Appl. Prob.9(1999)871-903

[11] W.H. Fleming and S.J. Sheu, "Risk-sensitive control and an optimal investment model" Mathematical Finance, 10 (2000)197-213

[12] W.H. Fleming and S.J. Sheu, "Risk-sensitive control and an optimal investment model (II)", preprint

[13] R.Z. Has'minskii, *Stochastic stability of differential equations*, Sijthoff and Noordhoff, Alphen aan den Rijin (1980)

[14] H. Kaise and H. Nagai, Bellman-Isaacs equations of ergodic type related to risk-sensitive control and their singular limits, *Asymptotic Analysis* 16 (1998) 347-362

[15] H. Kaise and H. Nagai, Ergodic type Bellman equations of risk-sensitive control with large parameters and their singular limits, *Asymptotic Analysis* 20 (1999) 279-299

[16] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York (1988)

[17] K.Kuroda and H. Nagai "Ergodic type Bellman equation of risk-sensitive control and potfolio optimization on infinite time horizon", a volume in honor of A. Benosussan, Eds. J.L. Menaldi et al. , IOS Press (2001) 530-538

[18] K.Kuroda and H. Nagai "Risk-sensitive portfolio optimization on infinite time horizon", (2000) Preprint

[19] O.A. Ladyzhenskaya and N.N. Ural'tseva, *" Linear and Quasi-linear Elliptic Equations"*, Academic Press, New York, 1968

[20] H. Nagai, Bellman equations of risk-sensitive control, *SIAM J. Control and Optimization*, Vol. 34, No. 1 (1996) 74-101.

[21] H. Nagai and S. Peng, Risk-sensitive dynamic portfolio optimization with partial information on infinite time horizon, to appear in Annals of Appl. Prob.

[22] D.W. Stroock and S.R.S. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, New York (1979)

[23] P. Whittle "Risk-sensitive linear quadratic Gaussian control", *Adv. Appl. Prob.*, vol. 13 (1982) 764-777

[24] W.M. Wonham, "On a Matrix Riccati Equation of Stochastic Control", *SIAM J. Control Optim.*, vol. 6 (1968) 681-697